Periodic Motion II.

Review on SHM

mechanical system obeys equation of motion

$$\frac{d^2x}{dt^2} = -\omega^2 x \quad , \quad f = \frac{\omega}{2\pi} \quad , \quad T = \frac{2\pi}{\omega}$$

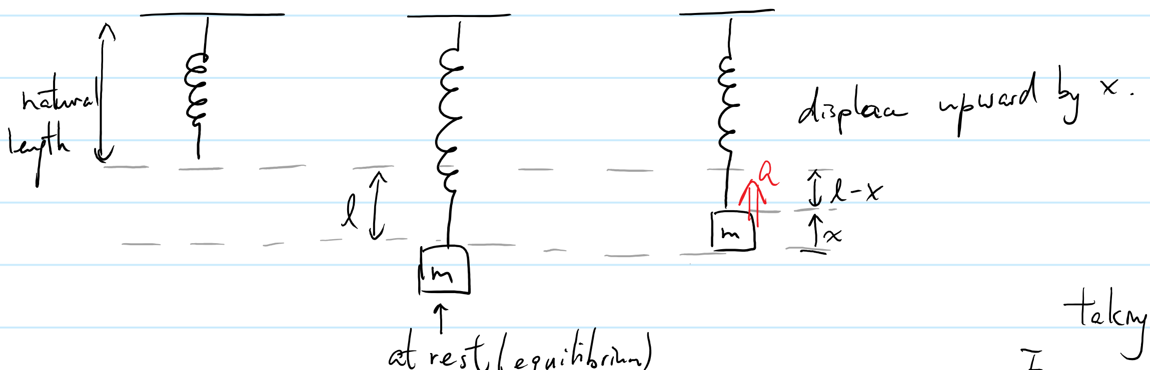
Solution: $x(t) = A\cos(\omega t + \phi)$ where A, ϕ are determined by initial condition (x_0, v_0)

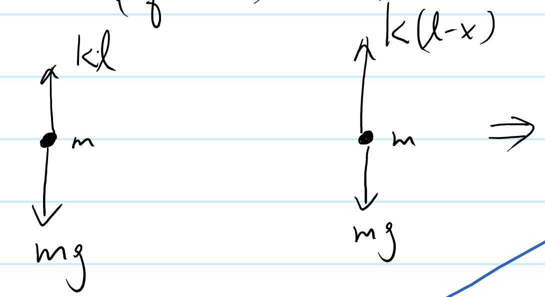
In general, x is a displacement from the equilibrium position

can be linear or angular

Some examples of SHM

- Vertical spring-mass system





$F = kl - mg = 0$

taking up as +ve.

$F = ma$

$k(l-x) - mg = ma$

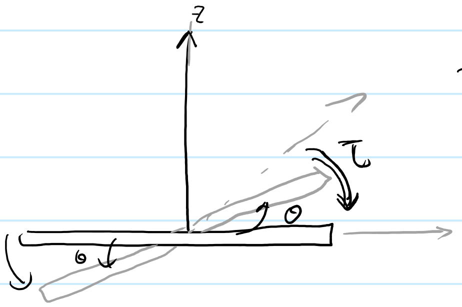
$\Rightarrow \overbrace{kl - mg}^{=0} - kx = ma$

$\Rightarrow ma = -kx$

$\Rightarrow \ddot{x} = a = -\frac{k}{m}x$ SHM

$\omega^2 = \frac{k}{m}$ same as horizontal spring.

Angular Oscillation



τ turning the rod back to original orientation.

$$\vec{\tau} \parallel -\hat{k}$$

$$\tau \propto \theta$$

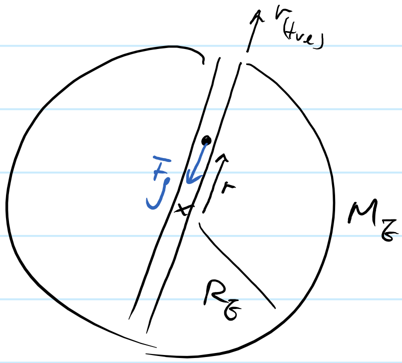
$$\tau_z = -K\theta \qquad \text{c.f. } F_{spring} = -kx$$

$$\tau_z = I\alpha \Rightarrow \qquad \alpha = \frac{d^2\theta}{dt^2} = \frac{\tau}{I} = -\underbrace{\frac{K}{I}}\theta$$

$$\omega^2 = \frac{K}{I}.$$

$$\text{Period} = T = 2\pi\sqrt{\frac{I}{K}} \sim 2\pi\sqrt{\frac{\text{Inertia}}{\text{restoring strength}}} \qquad \text{c.f. } T = 2\pi\sqrt{\frac{m}{k}}$$

- Tunnel through the Earth.



Recall the gravitational force inside a uniform solid sphere is

$$F_g = \frac{GM_E m}{R_E^3} \cdot r$$

A mass inside the tunnel through the Earth obeys.

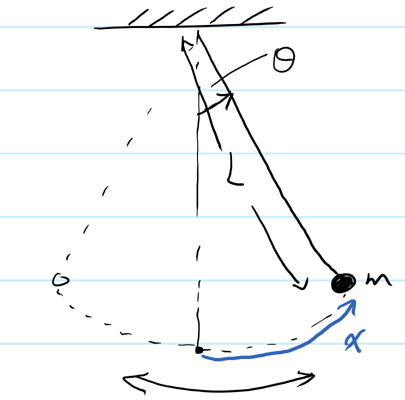
$$\vec{F} = m\vec{a}$$

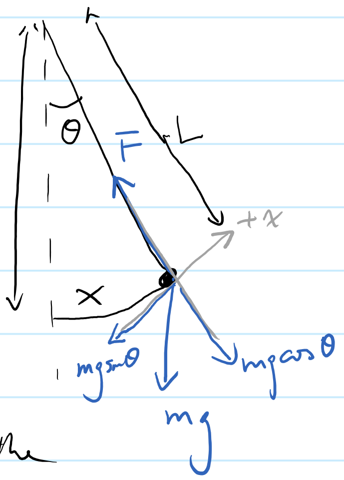
$$-\frac{GM_E m}{R_E^3} r = m\frac{d^2r}{dt^2}$$

$$\Rightarrow \quad \frac{d^2r}{dt^2} = -\underbrace{\boxed{\frac{GM_E}{R_E^3}}}_{\omega^2} \cdot r$$

$$T = 2\pi\sqrt{\frac{R_E^3}{GM_E}} \sim 1.5\ hr.$$

- Simple Pendulum





$x = L\theta$

Let x be the arc length shown in the figure.

The acceleration along x is due to only the component of mg along x; i.e, $mg\sin\theta$.

$$F = ma$$

$$\Rightarrow \quad -mg\sin\theta = m\frac{d^2x}{dt^2} \qquad\qquad x = L\theta \Rightarrow \frac{d^2x}{dt^2} = L\frac{d^2\theta}{dt^2}$$

$$\Rightarrow \quad -g\sin\theta = L\frac{d^2\theta}{dt^2} \quad \text{———} \quad (*)$$

for small θ ($\theta \lesssim 0.1$ rad or 5.7°)

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots$$
$$\approx \theta$$

$\theta^3, \theta^5 \dots$ very small compare to θ.

$(*)$ becomes

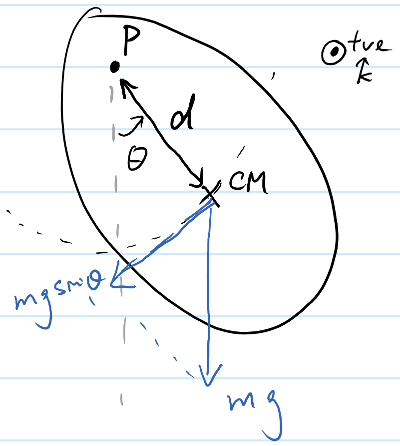
$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta \qquad \underline{\underline{\text{SHM}}}$$

$$\omega^2 = \frac{g}{L}, \quad T = 2\pi\sqrt{\frac{L}{g}}$$ independent of mass.

Physical Pendulum

Consider a rigid body pinned down on a wall at point P.

It can swing freely about P under gravity.



$$\vec{\tau} = -mg\sin\theta \cdot d \cdot \hat{k} = I_p \alpha\, \hat{k}$$

$$\Rightarrow \quad I_p \alpha = -mgd\sin\theta$$

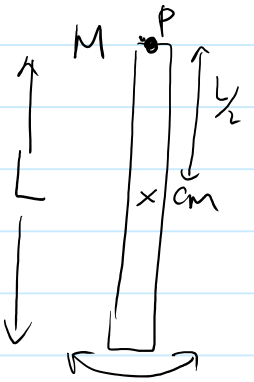
$$\alpha = \frac{d^2\theta}{dt^2} = -\frac{mgd}{I_p}\sin\theta \approx -\frac{mgd}{I_p}\theta$$

for small θ

$$\Rightarrow \quad \boxed{\frac{d^2\theta}{dt^2} = -\frac{mgd}{I_p}\theta} \quad \text{SHM.}$$

$$\omega^2 = \frac{mgd}{I_p}, \quad T = 2\pi\sqrt{\frac{I_p}{mgd}}$$

Example A rod swing about one of the ends

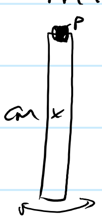


$$I_p = \frac{1}{3}ML^2$$

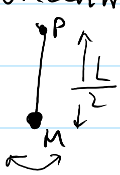
$$d = \frac{L}{2}$$

$$T = 2\pi\left(\frac{\frac{1}{3}ML^2}{Mg\frac{L}{2}}\right)^{\frac{1}{2}} = 2\pi\sqrt{\frac{2}{3}\frac{L}{g}}$$

Can I treat the rod as a simple pendulum with all the mass concentrated at its CM?



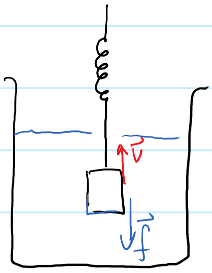
$\Rightarrow$



Simple pendulum $T = 2\pi\sqrt{\frac{\left(\frac{L}{2}\right)}{g}} = 2\pi\sqrt{\frac{L}{2g}} \neq 2\pi\sqrt{\frac{2}{3}\frac{L}{g}}$

$\Rightarrow$ Ans: NO! shape matters!

Damped Harmonic Oscillation



drag force : $\vec f = -b\vec v$ opposite to $\vec v$.

$$\vec F_{net} = \vec F_{spring} + \vec f = m\vec a$$

$$\Rightarrow \quad -bv - kx = ma \qquad v = \frac{dx}{dt}$$

$$\Rightarrow \quad \boxed{\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0}$$

eq^n of motion.

Solution : $x(t) = A e^{-\frac{b}{2m}t}\cos(\omega' t + \phi)$

$$\text{where} \quad \omega' = \sqrt{\underbrace{\frac{k}{m}} - \frac{b^2}{4m^2}} = \frac{1}{2m}\sqrt{(2\sqrt{km})^2 - b^2}$$

new angular frequency

old angular frequency $\sqrt{\frac{k}{m}} = \omega_0$

Practice : Try to verify $x(t)$ is the solution of the eq^n of motion.

Different values of k, m & b result in different ω' and different types of motion.

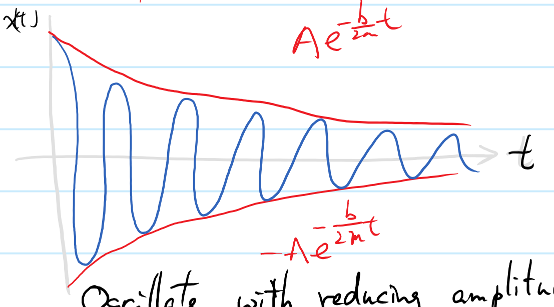
Case 1 (underdamp)

$b < 2\sqrt{km} \Rightarrow \omega' > 0$

$$x(t) = \underbrace{Ae^{-\frac{b}{2m}t}}\underbrace{\cos(\omega' t + \phi)} = A(t)\cos(\omega' t + \phi)$$

govern the maximum value of $x(t)$.

oscillate between $(-1, 1)$



Oscillate with reducing amplitude.

Case 2 (critical damp)

$b = 2\sqrt{km} \Rightarrow \omega' = 0$

$\Rightarrow$ No oscillations!

$x(t) = Ae^{-\frac{b}{2m}t}\cos(\phi) \propto e^{-\frac{b}{2m}t}$

Case 3 (overdamp)

$b > 2\sqrt{km} \Rightarrow \omega'$ is complex number!

$\Rightarrow$ No oscillations, too!

Energy

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

$$\frac{dE}{dt} = \frac{1}{2}m(2v)\dot{v} + \frac{1}{2}k(2x)\dot{x}$$

$$= mva + kxv$$

$$= (ma + kx)\,v \qquad \text{but } ma + kx + bv = 0$$

$$= -bv \cdot v \quad \Leftarrow \text{negative, energy loss.}$$

$$= \vec{f} \cdot \vec{v}$$

$$= \text{Power by drag force}$$

Forced oscillation

Consider to add a periodic driving force

$$F(t) = F_{max}\cos(\omega_d t) \qquad \omega_d \text{ is the driving freq.}$$

Eq$^{\underline{n}}$ of motion : $\quad ma = -kx - bv + F(t)$

Long time behaviour. $\quad x(t) \simeq A(\omega_d)\cos(\omega_d t + \phi)$
(for very large t)

the object oscillates with the driving freq. at an ω-dependent amplitude.

$$A(\omega_d) = \frac{F_{max}}{\sqrt{(k - m\omega_d^2)^2 + b^2\omega_d^2}} = \frac{F_{max}}{m} \cdot \frac{1}{\sqrt{\underbrace{(\omega_0^2 - \omega_d^2)^2}_{\text{goes to zero}} + \frac{b^2}{m^2}\omega_d^2}}$$

$A(\omega_d)$ is maximum when $\omega_d \simeq \omega_0$

This characteristic is called resonance.